Announcements

1) Reading for Wednesday:

Section 7.6

2) Official office hours

MW 9-9:30

M 1-2

T 2:30-3:30

Notation: (Riemann integral)

If f is Riemann integrable on $[a,b]$, we denote by

$$\boxed{\int_a^b f(x)\,dx}$$

the common value of $L(f)$ and $U(f)$.

Theorem: Let f be bounded on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if $\forall\, c \in (a,b)$, f is Riemann integrable on $[a,c]$ and $[c,b]$. Moreover,

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

proof: $\Rightarrow$) Suppose f is Riemann integrable on $[a,b]$. Let P be any partition of $[a,c]$, Q any partition of $[c,b]$.

$$U(f, P \cup Q) - L(f, P \cup Q)$$

$$\geq U(f,P) - L(f,P)$$

or

$$U(f,Q) - L(f,Q)$$

Let $\varepsilon > 0$. Let R be any partition of $[a,b]$ with

$$U(f,R) - L(f,R) < \varepsilon.$$

Let $T = R \cup \{c\}$.

Observe $U(f,T) - L(f,T) < \varepsilon$ since T refines R.

Let $P = \{x \in T \mid x \leq c\}$

$Q = \{x \in T \mid x \geq c\}$

Then $T = P \cup Q$,

$$\varepsilon > U(f,T) - L(f,T)$$

$$\geq U(f,P) - L(f,P)$$

or $U(f,Q) - L(f,Q)$

$\Rightarrow$ f is integrable on

$[a,c]$ and $[c,b]$.

$\Leftarrow$ Suppose f is integrable on $[a,c]$ and $[c,b]$.

Let $\varepsilon > 0$ and let P be a partition of $[a,c]$ with $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$.

Similarly, let Q be a partition of $[c,b]$ with

$$U(f,Q) - L(f,Q) < \frac{\varepsilon}{2}.$$

Let $R = P \cup Q$.

Then

$$U(f,R) - L(f,R)$$

$$= U(f,P) - L(f,P)$$
$$+ U(f,Q) - L(f,Q)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$$ so

f is integrable on $[a,b]$.

To show:
$$\int_a^b f(x)\,dx$$
$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Let P be any partition of $[a,b]$ and let $Q = P \cup \{c\}$.

If $P_1 = \{x \in Q \mid x \leq c\}$,
$P_2 = \{x \in Q \mid x \geq c\}$,
then

$$U(f,Q) = U(f,P_1) + U(f,P_2)$$

and $L(f,Q) = L(f,P_1) + L(f,P_2)$.

This implies that

$$\int_a^b f(x)\,dx \leq U(f,Q)$$
$$\leq U(f,P_1) + U(f,P_2)$$

$$\int_a^b f(x)\,dx \geq L(f,Q)$$
$$= L(f,P_1) + L(f,P_2)$$

We get

$$L(f,P_1)+L(f,P_2) \leq \int_a^b f(x)\,dx$$

$$\leq U(f,P_1)+U(f,P_2)$$

By choosing P_1, P_2 so that

$$U(f,P_i)-L(f,P_i) < \frac{\varepsilon}{2}$$

for $i \in \{1,2\}$, we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$\square$

Corollary: (finitely many discontinuities)

Suppose f has finitely many discontinuities on $[a,b]$. Then f is integrable on $[a,b]$.

Proof: Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be the discontinuities of f.

Let c be any point in

$$[x_{i-1}, x_i], \quad 1 < i \leq n.$$

We know that if

$x_{i-1} < a < c < b < x_i$,

f is continuous on

$[a, c]$ and $[c, b] \Rightarrow$

f is integrable on $[a, c]$ and

$[c, b]$. Use theorem from

last class to conclude f is

integrable on $[x_{i-1}, c]$

and $[c, x_i]$.

By previous theorem, f is then integrable on $[x_{i-1}, x_i]$ $\forall\, i$, $1 < i \leq n$. Applying the theorem again yields f is integrable on $[a,b]$.

Moreover,

$$\int_a^b f(x)dx = \sum_{i=2}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx \right) \quad \square$$

Theorem (integration properties)

Let f_1, f_2 be integrable on $[a,b]$. Then

a) $f_1 + f_2$ is integrable on $[a,b]$ and

$$\int_a^b f_1 + f_2 \, dx = \int_a^b f_1 \, dx + \int_a^b f_2 \, dx$$

b) If $k \in \mathbb{R}$, kf_1 is integrable on $[a,b]$, and

$$k\int_a^b f_1(x)\,dx = \int_a^b k f_1(x)\,dx$$

c) If $f_1 \le f_2$ on $[a,b]$,

then $\displaystyle\int_a^b f_1(x)\,dx \le \int_a^b f_2(x)\,dx$

d) We have $|f_1|$ is integrable

on $[a,b]$ and

$$\left| \int_a^b f_1(x)\,dx \right| \le \int_a^b |f_1(x)|\,dx$$

Proof: c) If $f_1 \le f_2$,

then

$$U(f_1, P) \le U(f_2, P)$$

$\forall$ partitions P of $[a,b]$.

Taking infima (almost),

$$\int_a^b f_1(x)\,dx \le \int_a^b f_2(x)\,dx.$$

Definition: (measure zero)

A subset $S \subseteq \mathbb{R}$ has (Lebesgue) measure zero if $\forall\ \varepsilon > 0$, $\exists$ a countable collection $\{O_i\}_{i=1}^{\infty}$ of open intervals with $\boxed{S \subseteq \bigcup_{i=1}^{\infty} O_i}$

and $\boxed{\sum_{i=1}^{\infty} \ell(O_i) < \varepsilon}$

($\ell(O_i)$ = length of O_i)

Example: 1) Any finite set has measure zero.

2) Any countable set has measure zero.

3) The Cantor set has measure zero.

Theorem: (Lebesgue) Let f be bounded on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if the set of discontinuities of f has measure zero.

Proof: Starting Wednesday.